

Well-posedness of the Cauchy problem for a space-dependent anyon Boltzmann equation.

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Abstract. A fully non-linear kinetic Boltzmann equation for anyons is studied in a periodic 1d setting with large initial data. Strong L^1 solutions are obtained for the Cauchy problem. The main results concern global existence, uniqueness and stability.

1 Anyons and the Boltzmann equation.

Let us first recall the definition of anyon. Consider the wave function $\psi(R, \theta, r, \varphi)$ for two identical particles with center of mass coordinates (R, θ) and relative coordinates (r, φ) . Exchanging them, $\varphi \rightarrow \varphi + \pi$, gives a phase factor $e^{2\pi i}$ for bosons and $e^{\pi i}$ for fermions. In three or more dimensions those are all possibilities. Leinaas and Myrheim proved in 1977 [9], that in one and two dimensions any phase factor is possible in the particle exchange. This became an important topic after the first experimental confirmations in the early 1980-ies, and Frank Wilczek in analogy with the terms bos(e)-ons and fermi-ons coined the name any-ons for the new quasi-particles with any phase. Anyon quasi-particles with e.g. fractional electric charge, have since been observed in various types of experiments.

By moving to a definition in terms of a generalized Pauli exclusion principle, Haldane [8] extended this to a fractional exclusion statistics valid for any dimension, and coinciding with the anyon definition in the one and two dimensional cases. Haldane statistics has also been realized for neutral fermionic atoms at ultra-low temperatures in three dimensions [3]. Wu later derived [17] occupation-number distributions for ideal gases under Haldane statistics by counting states under the new fractional exclusion principle. From the number of quantum states of N identical particles occupying G states being

$$\frac{(G + N - 1)!}{N!(G - 1)!} \quad \text{and} \quad \frac{G!}{N!(G - N)!}$$

in the boson resp. fermion cases, he derived the interpolated number of quantum states for the fractional exclusions to be

$$\frac{(G + (N - 1)(1 - \alpha))!}{N!(G - \alpha N - (1 - \alpha))!}, \quad 0 < \alpha < 1. \quad (1.1)$$

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He then obtained for ideal gases the equilibrium statistical distribution

$$\frac{1}{w(e^{(\epsilon-\mu)/T}) + \alpha}, \quad (1.2)$$

where ϵ denotes particle energy, μ chemical potential, T temperature, and the function $w(\zeta)$ satisfies

$$w(\zeta)^\alpha (1 + w(\zeta))^{1-\alpha} = \zeta \equiv e^{(\epsilon-\mu)/T}.$$

In particular $w(\zeta) = \zeta - 1$ for $\alpha = 0$ (bosons) and $w(\zeta) = \zeta$ for $\alpha = 1$ (fermions).

In elastic pair collisions, the velocities (v, v_*) before and (v', v'_*) after a collision are related by

$$v' = v - n[(v - v_*) \cdot n], \quad v'_* = v_* + n[(v - v_*) \cdot n], \quad n \in S^{d-1}.$$

This preserves mass, linear momentum, and energy in Boltzmann type collision operators. We shall write $f = f(v)$, $f_* = f(v_*)$, $f' = f(v')$, $f'_* = f(v'_*)$. An important question for gases with fractional exclusion statistics, is how to calculate their transport properties, in particular how the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f)$$

gets modified. An answer was given by Bhaduri, Bhalerao, and Murthy [2] by generalizing to anyons the filling factors $F(f)$ from the fermion and boson cases, $F(f) = (1 + \eta f)$, $\eta = \mp 1$, and by inductive reasoning obtaining as anyon filling factors $F(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha)f)^{1-\alpha}$, $0 < \alpha < 1$.

Namely, with a filling factor $F(f)$ in the collision operator Q , the entropy production term becomes

$$\int Q(f) \log \frac{f}{F(f)} dv,$$

which for equilibrium implies

$$\frac{f'}{F(f')} \frac{f'_*}{F(f'_*)} = \frac{f}{F(f)} \frac{f_*}{F(f_*)}.$$

Using conservation laws and properties of the Cauchy equation, one concludes that in equilibrium $\frac{f}{F(f)}$ is a Maxwellian. Inserting Wu's equilibrium (1.2) for f and taking the quotient Maxwellian as $e^{-(\epsilon-\mu)/T}$, this gives

$$f = \frac{1}{w(e^{(\epsilon-\mu)/T}) + \alpha}, \quad F(f) = f e^{(\epsilon-\mu)/T} = \frac{e^{(\epsilon-\mu)/T}}{w(e^{(\epsilon-\mu)/T}) + \alpha}.$$

In particular in the fermion and boson cases,

$$f = \frac{1}{e^{(\epsilon-\mu)/T} - \eta}, \quad F(f) = \frac{e^{(\epsilon-\mu)/T}}{e^{(\epsilon-\mu)/T} - \eta}, \quad \eta = \mp 1.$$

This is consistent with taking an interpolation between the fermion and boson factors as general filling factor, $F(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha)f)^{1-\alpha}$, $0 < \alpha < 1$. It gives the collision operator Q of [2] for Haldane statistics,

$$Q(f)(v) = Q^+(f) - Q^-(f) = \int_{\mathbb{R}^d \times S^{d-1}} B(|v - v_*|, \omega) [f' f'_* F(f) F(f_*) - f f_* F(f') F(f'_*)] dv_* d\omega. \quad (1.3)$$

Here $d\omega$ corresponds to the Lebesgue probability measure on the $(d-1)$ -sphere. The collision kernel $B(z, \omega)$ in the variables $(z, \omega) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$ is positive, locally integrable, and only depends on $|z|$

and $|(z, \omega)|$. See [2] for a further discussion of the kernel B .

The anyon Boltzmann equation for $0 < \alpha < 1$ retains important properties from the Fermi-Dirac case, but it has so far not been validated from basic quantum theory. In the filling factor $F(f) = (1 - \alpha f)^\alpha (1 + (1 - \alpha)f)^{1-\alpha}$, $0 < \alpha < 1$, the factor $(1 - \alpha f)^\alpha$ requires the value of f to be between 0 and $\frac{1}{\alpha}$. This is formally preserved by the equation, since the gain term vanishes for $f = \frac{1}{\alpha}$, making the Q -term (1.3) and the derivative left hand side of the Boltzmann equation negative there. And the derivative equals the positive gain term for $f = 0$, where the loss term vanishes. F is concave with maximum value one at $f = 0$ for $\alpha \geq \frac{1}{2}$, and maximum value $(\frac{1}{\alpha} - 1)^{1-2\alpha} > 1$ at $f = \frac{1-2\alpha}{\alpha(1-\alpha)}$ for $\alpha < \frac{1}{2}$. The collision operator vanishes identically for the equilibrium distribution functions obtained by Wu, but for no other functions.

The Boltzmann equation for the limiting cases, representing boson statistics ($\alpha = 0$) and fermion statistics ($\alpha = 1$), was introduced by Nordheim [15] in 1928. Here the quartic terms in the collision integral cancel, which is used in the analysis. General existence results for the space-homogeneous isotropic boson large data case were obtained in [11], followed by a number of other papers, e.g. [7], [12], [13], [14], and for the space-dependent case near equilibrium in [16]. In the space-dependent fermion case general existence results were obtained in [6] and [10].

For $0 < \alpha < 1$ there are no cancellations in the collision term. Moreover, the Lipschitz continuity of the collision term is replaced by a weaker Hölder continuity near $f = \frac{1}{\alpha}$. The space-homogeneous initial value problem for the Boltzmann equation with Haldane statistics is

$$\frac{df}{dt} = Q(f), \quad f(0, v) = f_0(v). \quad (1.4)$$

Because of the filling factor F , the range for the initial value f_0 should belong to $[0, \frac{1}{\alpha}]$, which is also formally preserved by the equation. A good control of $\int f(t, x, v) dv$, which in the space-homogeneous case is given by the mass conservation, can be used to keep f uniformly away from $\frac{1}{\alpha}$, and $F(f)$ Lipschitz continuous. That was a basic observation behind the existence result for the space-homogeneous anyon Boltzmann equation.

Proposition 1.1 [1] *Consider the space-homogeneous equation (1.4) with velocities in \mathbb{R}^d , $d \geq 2$ and for hard force kernels with*

$$0 < B(z, \theta) \leq C|z|^\beta |\sin \theta \cos \theta|^{d-1}, \quad (1.5)$$

where $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, $0 < \beta \leq 1$, $d > 2$, and $0 < \beta < 1$, $d = 2$. Let the initial value f_0 have finite mass and energy. If $0 < f_0 \leq \frac{1}{\alpha}$ and $\text{ess sup}(1 + |v|^s) f_0 < \infty$ for $s = d - 1 + \beta$, then the initial value problem for (1.4) has a strong solution in the space of functions continuous from $t \geq 0$ into $L^1 \cap L^\infty$, which conserves mass and energy, and for $t_0 > 0$ given, has $\text{ess sup}_{v, t \leq t_0} |v|^{s'} f(t, v)$ bounded, where $s' = \min(s, \frac{2\beta(d+1)+2}{d})$.

In this proposition, stronger limitations on B would allow for weaker conditions on the initial value f_0 . The proof implies stability; given a sequence of positive initial values $(f_{0n})_{n \in \mathbb{N}}$ with

$$\sup_n \text{ess sup} f_{0n}(v) < \frac{1}{\alpha},$$

and converging in L^1 to f_0 , there is a subsequence of the solutions converging in L^1 to a solution with initial value f_0 .

2 The main results.

The present paper considers the space-dependent anyon Boltzmann equation in a slab. Anyons only exist in one and two dimensions. The proof in this paper uses an estimate for the Bony functional in one space dimension, which due to the filling factor $F(f)$, is restricted to the anyon case $v \in \mathbb{R}^2$. For $\cos\theta = n \cdot \frac{v-v_*}{|v-v_*|}$, the kernel $B(|v-v_*|, \theta)$ is assumed measurable with

$$0 \leq B \leq B_0, \quad (2.1)$$

for some $B_0 > 0$. It is also assumed for some $\gamma, \gamma', c_B > 0$, that

$$B(|v-v_*|, \theta) = 0 \text{ for } |\cos\theta| < \gamma', \quad \text{for } 1 - |\cos\theta| < \gamma', \quad \text{and for } |v-v_*| < \gamma, \quad (2.2)$$

and that

$$\int B(|v-v_*|, \theta) d\theta \geq c_B > 0 \quad \text{for } |v-v_*| \geq \gamma. \quad (2.3)$$

The initial datum $f_0(x, v)$, periodic in x , is assumed to be a measurable function with values in $[0, \frac{1}{\alpha}]$, and such that

$$(1 + |v|^2)f_0(x, v) \in L^1([0, 1] \times \mathbb{R}^2), \quad \int \sup_{x \in [0, 1]} f_0(x, v) dv = c_0 < \infty, \quad \inf_{x \in [0, 1]} f_0(x, v) > 0, \quad \text{a.a. } v \in \mathbb{R}^2. \quad (2.4)$$

With v_1 denoting the component of v in the x -direction, consider for functions periodic in x , the initial value problem

$$\partial_t f(t, x, v) + v_1 \partial_x f(t, x, v) = Q(f)(t, x, v), \quad f(0, x, v) = f_0(x, v), \quad (t, x, v) \in \mathbb{R}_+ \times [0, 1] \times \mathbb{R}^2. \quad (2.5)$$

The main result of the present paper is the following theorem.

Theorem 2.1

There exists a strong solution $f \in \mathcal{C}([0, \infty[; L^1([0, 1] \times \mathbb{R}^2))$ of (2.5) with $0 < f(t, \cdot) < \frac{1}{\alpha}$ for $t > 0$. There is $t_m > 0$ such that for any $T > t_m$, there is $\eta_T > 0$ so that $f \leq \frac{1}{\alpha} - \eta_T$ for $t_m \leq t \leq T$. The solution is unique and stable in the L^1 -norm on each interval of time $[0, T]$. It conserves mass, first v -moments and energy.

Remarks.

The above results seem to be new also in the fermion case where $\alpha = 1$.

The approach in the paper can also be used to obtain regularity results.

The control of $\int f(t, x, v) dv$ in the present space-dependent setting is non-trivial.

The asymptotic behaviour of the solution, not considered in this paper, is related to an entropy for (2.5),

$$\int \left(f \log f + \left(\frac{1}{\alpha} - f \right) \log(1 - \alpha f)^\alpha - \left(\frac{1}{1 - \alpha} + f \right) \log(1 + (1 - \alpha)f)^{1 - \alpha} \right) dx dp.$$

An open problem is the behaviour of (2.5) beyond the anyon frame, i.e. for higher v -dimensions under Haldane statistics. It seems likely that a close to equilibrium approach as in the classical case, could work with fairly general kernels B for close to equilibrium initial values f_0 with some regularity and strong decay conditions for large velocities. Any progress on the large data case in

several space-dimensions under Haldane statistics would be quite interesting.

The lack of Lipschitz continuity of $F(f)$ when f is in a neighborhood of $\frac{1}{\alpha}$ requires some care. Since the gain term vanishes when $f = \frac{1}{\alpha}$ and the derivative becomes negative there, f should start decreasing before reaching this value. The proof that this takes place uniformly over phase-space and approximations, is based on a good control of $\int f(t, x, v) dv$ in the integration of the gain and loss parts of Q . That is a main topic in Section 3 together with the study of a family of approximating equations with large velocity cut-off. Based on those results and using the Lipschitz continuity of $F(\cdot)$ away from $\frac{1}{\alpha}$, in Section 4 contraction mapping techniques prove the well-posedness of the problem, when the initial value f_0 stays uniformly away from $\frac{1}{\alpha}$. That restriction is removed by a local initial value analysis, which only assumes Hölder continuity of $F(\cdot)$.

3 Approximations and control of mass density.

For any $j \in \mathbb{N}^*$, denote by ψ_j , the cut-off function with

$$\psi_j(r) = 0 \quad \text{if } r > j \quad \text{and} \quad \psi_j(r) = 1 \quad \text{if } r \leq j,$$

and set

$$\chi_j(v, v_*, v', v'_*) = \psi_j(|v|)\psi_j(|v_*|)\psi_j(|v'|)\psi_j(|v'_*|).$$

Let F_j be defined on $[0, \frac{1}{\alpha}]$ by

$$F_j(y) = \frac{1 - \alpha y}{(\frac{1}{j} + 1 - \alpha y)^{1-\alpha}} (1 + (1 - \alpha)y)^{1-\alpha}.$$

Denote by Q_j (resp. Q_j^+), the operator

$$\begin{aligned} Q_j(f)(v) &:= \frac{1}{\pi} \int B(|v - v_*|, \theta) \chi_j(v, v_*, v', v'_*) \left(f' f'_* F_j(f) F_j(f_*) - f f_* F_j(f') F_j(f'_*) \right) dv_* d\theta, \\ (\text{resp. its gain part } Q_j^+(f)(v) &:= \frac{1}{\pi} \int B(|v - v_*|, \theta) \chi_j(v, v_*, v', v'_*) f' f'_* F_j(f) F_j(f_*) dv_* d\theta). \end{aligned}$$

For $j \in \mathbb{N}^*$, let a mollifier φ_j be defined by $\varphi_j(x, v) = j^3 \varphi(jx, jv)$, where

$$\varphi \in C_0^\infty(\mathbb{R}^3), \quad \text{support}(\varphi) \subset [0, 1] \times \{v \in \mathbb{R}^2; |v| \leq 1\}, \quad \varphi \geq 0, \quad \int \varphi(x, v) dx dv = 1.$$

Let $f_{0,j}$ be the restriction to $[0, 1] \times \{v; |v| \leq j\}$ of $(\min\{f_0, \frac{1}{\alpha} - \frac{1}{j}\}) * \varphi_j$. The following lemma concerns a corresponding approximation of (2.5).

Lemma 3.1 *For $T > 0$, there is a unique solution $f_j \in C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\}))$ to*

$$\partial_t f_j + v_1 \partial_x f_j = Q_j(f_j), \quad f_j(0, \cdot, \cdot) = f_{0,j}, \tag{3.1}$$

with values in $]0, \frac{1}{\alpha} - \eta_j]$, for some $\eta_j > 0$. It conserves mass, first moment and energy.

Proof of Lemma 3.1.

Let $T > 0$ be given. We shall first prove by contraction that for $T_1 > 0$ and small enough, there is a unique solution

$$f_{\epsilon,j} \in C([0, T_1] \times [0, 1]; L^1(\{v; |v| \leq j\})) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$$

to (3.1). Let the map \mathcal{C} be defined on periodic in x functions in $C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\})) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$ by $\mathcal{C}(f) = g$, where

$$\partial_t g + v_1 \partial_x g = \frac{1}{\pi} (1 - \alpha g) \left(\frac{1 + (1 - \alpha)f}{\frac{1}{j} + 1 - \alpha f} \right)^{1-\alpha} \int B \chi_j f' f'_* F_j(f_*) dv_* d\theta - \frac{g}{\pi} \int B \chi_j f_* F_j(f') F_j(f'_*) dv_* d\theta,$$

$$g(0, \cdot, \cdot) = f_{0,j}.$$

It follows from the linearity of the previous partial differential equation that it has a unique periodic solution g in $C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\}))$. For f with values in $[0, \frac{1}{\alpha}]$, g takes its values in $]0, \frac{1}{\alpha}]$. Indeed, denoting by $g^\sharp(t, x, v) = g(t, x + tv_1, v)$,

$$g^\sharp(t, x, v) \geq f_{0,j}(x + tv_1, v) e^{-\int_0^t \bar{\sigma}_f^\sharp(r, x, v) dr} > 0,$$

and

$$\begin{aligned} (1 - \alpha g)^\sharp(t, x, v) &= (1 - \alpha f_{0,j})(x + tv_1, v) e^{-\int_0^t \bar{\sigma}_f^\sharp(r, x, v) dr} \\ &\quad + \frac{\alpha}{\pi} \int_0^t \left(g \int B \chi_j f_* F_j(f') F_j(f'_*) dv_* d\theta \right)^\sharp(s, x, v) e^{-\int_s^t \bar{\sigma}_f^\sharp(r, x, v) dr} ds \\ &\geq (1 - \alpha f_{0,j})(x + tv_1, v) e^{-\int_0^t \bar{\sigma}_f^\sharp(r, x, v) dr} \geq 0. \end{aligned}$$

Here,

$$\begin{aligned} \bar{\sigma}_f &:= \frac{\alpha}{\pi} \left(\frac{1 + (1 - \alpha)f}{\frac{1}{j} + 1 - \alpha f} \right)^{1-\alpha} \int B \chi_j f' f'_* F_j(f_*) dv_* d\theta + \frac{1}{\pi} \int B \chi_j f_* \tilde{F}_{\epsilon,j}(f') \tilde{F}_{\epsilon,j}(f'_*) dv_* d\theta, \\ \tilde{\sigma}_f &:= \frac{\alpha}{\pi} \left(\frac{1 + (1 - \alpha)f}{\frac{1}{j} + 1 - \alpha f} \right)^{1-\alpha} \int B \chi_j f' f'_* F_j(f_*) dv_* d\theta. \end{aligned}$$

\mathcal{C} is a contraction on $C([0, T_1] \times [0, 1]; L^1(\{v; |v| \leq j\})) \cap \{f; f \in [0, \frac{1}{\alpha}]\}$, for $T_1 > 0$ small enough only depending on j , since the derivative of the map F_j is bounded on $[0, \frac{1}{\alpha}]$. Let f_j be its fixed point, i.e. the solution of (3.1) on $[0, T_1]$. The argument can be repeated and the solution can be continued up to $t = T$. By the exponential form for f_j (resp. $1 - \alpha f_j$)

$$f_j^\sharp(t, x, v) \geq f_{0,j}(x, v) e^{-\int_0^t \bar{\sigma}_{f_j}^\sharp(r, x, v) dr} > 0, \quad t \in [0, T], \quad x \in [0, 1], \quad |v| \leq j,$$

(resp.

$$\begin{aligned} (1 - \alpha f_j)^\sharp(t, x, v) &\geq (1 - \alpha f_{0,j})(x + tv_1, v) e^{-\int_0^t \bar{\sigma}_{f_j}^\sharp(r, x, v) dr} \\ &\geq \frac{1}{j e^{cj^3 T}}, \quad t \in [0, T], \quad x \in [0, 1], \quad |v| \leq j). \end{aligned}$$

Consequently, for some $\eta_j > 0$, there is a periodic in x solution $f_j \in C([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\}))$ to (3.1) with values in $]0, \frac{1}{\alpha} - \eta_j]$.

If there were another nonnegative local solution \tilde{f}_j to (3.1), defined on $[0, T']$ for some $T' \in]0, T]$, then by the exponential form it would stay below $\frac{1}{\alpha}$. The difference $f_j - \tilde{f}_j$ would for some constant $c_{T'}$ satisfy

$$\int |(f_j - \tilde{f}_j)^\sharp(t, x, v)| dx dv \leq c_{T'} \int_0^t |(f_j - \tilde{f}_j)^\sharp(s, x, v)| ds dx dv, \quad t \in [0, T'], \quad (f_j - \tilde{f}_j)^\sharp(0, x, v) = 0,$$

implying that the difference would be identically zero on $[0, T']$. Thus f_j is the unique solution on $[0, T]$ to (3.1), and has its range contained in $]0, \frac{1}{\alpha} - \eta_j]$.

Moreover, $f_j \in W^{1,1}([0, T] \times [0, 1]; L^1(\{v; |v| \leq j\}))$. Indeed, $\partial_x f_j$ satisfies

$$\begin{aligned} \partial_t(\partial_x f_j) + v_1 \partial_x(\partial_x f_j) + \sigma_j \partial_x f_j &= \frac{1}{\pi} (1 - \alpha f_j) \partial_x \left(\left(\frac{1 + (1 - \alpha f_j)}{\frac{1}{j} + 1 - \alpha f_j} \right)^{1-\alpha} \int B \chi_j f'_j f'_{j*} F_j(f_{j*}) dv_* d\theta \right) \\ &\quad - \frac{f_j}{\pi} \partial_x \int B \chi_j f_{j*} F'_j(f'_j) F_j(f'_{j*}) dv_* d\theta, \end{aligned} \quad (3.2)$$

$$\partial_x f_j(0, \cdot, \cdot) = \partial_x f_{0,j}, \quad (3.3)$$

where

$$\sigma_j := \frac{\alpha}{\pi} \left(\frac{1 + (1 - \alpha f_j)}{\frac{1}{j} + 1 - \alpha f_j} \right)^{1-\alpha} \int B \chi_j f'_j f'_{j*} F_j(f_{j*}) dv_* d\theta + \frac{1}{\pi} \int B \chi_j f_{j*} F'_j(f'_j) F_j(f'_{j*}) dv_* d\theta.$$

Using the exponential form of $\partial_x f_j$, multiplying it by $\text{sgn}(\partial_x f_j)$, integrating the resulting equation on $[0, 1] \times \{v; |v| \leq j\}$ and using a Gronwall argument leads to a j -dependent bound for $\int |\partial_x f_j(t, x, v)| dx dv$ on $[0, T]$. Hence, also from (3.1) and the bounded domain of integration of v , $\partial_t f_j$ also belongs to $L^\infty(0, T; L^1([0, 1] \times \{v; |v| \leq j\}))$. \blacksquare

The remaining part of this section is devoted to obtaining a uniform control with respect to $j \in \mathbb{N}^*$ of

$$\int \sup_{t>0, x \in [0, 1]} f_j^\sharp(t, x, v) dv.$$

It relies on the following four lemmas, where the first is an estimate of the Bony functionals,

$$\bar{B}_j(t) := \int_0^1 \int |v - v_*|^2 B \chi_j f_j f_{j*} F'_j(f'_j) F_j(f'_{j*}) dv dv_* d\theta dx, \quad t \geq 0.$$

Lemma 3.2

For $T > 0$ it holds that

$$\int_0^T \bar{B}_j(t) dt \leq c'_0(1 + T), \quad j \in \mathbb{N}^*,$$

with c'_0 only depending on $\int f_0(x, v) dx dv$ and on $\int |v|^2 f_0(x, v) dx dv$.

Proof of Lemma 3.2.

Denote f_j by f for simplicity. The proof is an extension of the classical one (cf [4], [5]), as follows. The integral over time of the momentum $\int v_1 f(t, 0, v) dv$ (resp. the momentum flux

$\int v_1^2 f(t, 0, v) dv$) is first controlled. Let $\beta \in C^1([0, 1])$ be such that $\beta(0) = -1$ and $\beta(1) = 1$. Multiply (3.1) by $\beta(x)$ (resp. $v_1 \beta(x)$) and integrate over $[0, t] \times [0, 1] \times \mathbb{R}^2$. It gives

$$\begin{aligned} \int_0^t \int v_1 f(\tau, 0, v) dv d\tau &= \frac{1}{2} \left(\int \beta(x) f_0(x, v) dx dv - \int \beta(x) f(t, x, v) dx dv \right. \\ &\quad \left. + \int_0^t \int \beta'(x) v_1 f(\tau, x, v) dx dv d\tau \right), \end{aligned}$$

(resp.

$$\begin{aligned} \int_0^t \int v_1^2 f(\tau, 0, v) dv d\tau &= \frac{1}{2} \left(\int \beta(x) v_1 f_0(x, v) dx dv - \int \beta(x) v_1 f(t, x, v) dx dv \right. \\ &\quad \left. + \int_0^t \int \beta'(x) v_1^2 f(\tau, x, v) dx dv d\tau \right). \end{aligned}$$

Consequently, using the conservation of mass and energy of f ,

$$|\int_0^t \int v_1 f(\tau, 0, v) dv d\tau| + \int_0^t \int v_1^2 f(\tau, 0, v) dv d\tau \leq c(1 + t). \quad (3.4)$$

Let

$$\mathcal{I}(t) = \int_{x < y} (v_1 - v_{*1}) f(t, x, v) f(t, y, v_*) dx dy dv dv_*.$$

It results from

$$\mathcal{I}'(t) = - \int (v_1 - v_{*1})^2 f(t, x, v) f(t, x, v_*) dx dv dv_* + 2 \int v_{*1} (v_{*1} - v_1) f(t, 0, v_*) f(t, x, v) dx dv dv_*,$$

and the conservations of the mass, momentum and energy of f that

$$\begin{aligned} &\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(s, x, v) f_*(s, x, v_*) dv dv_* dx ds \\ &\leq 2 \int f_0(x, v) dx dv \int |v_1| f_0(x, v) dv + 2 \int f(t, x, v) dx dv \int |v_1| f(t, x, v) dx dv \\ &+ 2 \int_0^t \int v_{*1} (v_{*1} - v_1) f(\tau, 0, v_*) f(\tau, x, v) dx dv dv_* d\tau \\ &\leq 2 \int f_0(x, v) dx dv \int (1 + |v|^2) f_0(x, v) dv + 2 \int f(t, x, v) dx dv \int (1 + |v|^2) f(t, x, v) dx dv \\ &+ 2 \int_0^t \left(\int v_{*1}^2 f(\tau, 0, v_*) dv_* \right) d\tau \int f_0(x, v) dx dv - 2 \int_0^t \left(\int v_{*1} f(\tau, 0, v_*) dv_* \right) d\tau \int v_1 f_0(x, v) dx dv \\ &\leq c \left(1 + \int_0^t \int v_1^2 f(\tau, 0, v) dv d\tau + \left| \int_0^t \int v_1 f(\tau, 0, v) dv \right| \right). \end{aligned}$$

And so, by (3.4),

$$\int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f(\tau, x, v) f(\tau, x, v_*) dx dv dv_* d\tau \leq c(1 + t). \quad (3.5)$$

Here, c is a constant depending only on $\int f_0(x, v) dx dv$ and $\int |v|^2 f_0(x, v) dx dv$.

Denote by $u_1 = \frac{\int v_1 f dv}{\int f dv}$. It holds

$$\begin{aligned}
& \int_0^t \int_0^1 \int (v_1 - u_1)^2 B \chi_j f f_* F_j(f') F_j(f'_*)(s, x, v, v_*, \theta) dv dv_* d\theta dx ds \\
& \leq c \int_0^t \int_0^1 \int (v_1 - u_1)^2 f f_*(s, x, v, v_*) dv dv_* dx ds \\
& = \frac{c}{2} \int_0^t \int_0^1 \int (v_1 - v_{*1})^2 f f_*(s, x, v, v_*) dv dv_* dx ds \\
& \leq c(1 + t).
\end{aligned} \tag{3.6}$$

Multiply equation (3.1) for f by v_1^2 , integrate and use that $\int v_1^2 Q_j(f) dv = \int (v_1 - u_1)^2 Q_j(f) dv$ and (3.6). It results

$$\begin{aligned}
& \frac{1}{\pi} \int_0^t \int (v_1 - u_1)^2 B \chi_j f' f'_* F_j(f) F_j(f'_*) dv dv_* d\theta dx ds \\
& = \int v_1^2 f(t, x, v) dx dv - \int v_1^2 f_0(x, v) dx dv + \frac{1}{\pi} \int_0^t \int (v_1 - u_1)^2 B \chi_j f f_* F_j(f') F_j(f'_*) dx dv dv_* d\theta ds \\
& < c_0(1 + t),
\end{aligned}$$

where c_0 is a constant only depending on $\int f_0(x, v) dx dv$ and $\int |v|^2 f_0(x, v) dx dv$.

After a collision transform the left hand side can be written

$$\begin{aligned}
& \frac{1}{\pi} \int_0^t \int (v'_1 - u_1)^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& = \frac{1}{\pi} \int_0^t \int (c_1 - n_1[(v - v_*) \cdot n])^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds,
\end{aligned}$$

where $c_1 = v_1 - u_1$. Expand $(c_1 - n_1[(v - v_*) \cdot n])^2$, and remove the positive term containing c_1^2 .

The term containing $n_1^2[(v - v_*) \cdot n]^2$ is estimated from below. When n is replaced by an orthogonal (direct) unit vector n_\perp , v' and v'_* are shifted and the product $f f_* F_j(f') F_j(f'_*)$ is unchanged. In \mathbb{R}^2 the ratio between the sum of the integrand factors $n_1^2[(v - v_*) \cdot n]^2 + n_{\perp 1}^2[(v - v_*) \cdot n_\perp]^2$ and $|v - v_*|^2$, is, outside of the angular cut-off (2.2), uniformly bounded from below by γ'^2 . Indeed, if α denotes the angle between $\frac{v - v_*}{|v - v_*|}$ and n ,

$$\begin{aligned}
n_1^2 \left[\frac{v - v_*}{|v - v_*|} \cdot n \right]^2 + n_{\perp 1}^2 \left[\frac{v - v_*}{|v - v_*|} \cdot n_\perp \right]^2 &= \cos^2 \theta \cos^2 \alpha + \sin^2 \theta \sin^2 \alpha \\
&\geq \gamma'^2 \cos^2 \alpha + \gamma'(2 - \gamma') \sin^2 \alpha \\
&\geq \gamma'^2, \quad \gamma' < |\cos \theta| < 1 - \gamma', \quad \alpha \in [0, 2\pi].
\end{aligned}$$

This is where the condition $v \in \mathbb{R}^2$ is used.

That leads to the lower bound

$$\begin{aligned}
& \int_0^t \int n_1^2[(v - v_*) \cdot n]^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& \geq \gamma'^2 \int_0^t \int |v - v_*|^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds.
\end{aligned}$$

And so,

$$\begin{aligned}
& \gamma'^2 \int_0^t \int |v - v_*|^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& \leq c_0(1+t) + 2 \int_0^t \int (v_1 - u_1) n_1 [(v - v_*) \cdot n] B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& \leq c_0(1+t) + 2 \int_0^t \int \left(v_1(v_2 - v_{*2}) n_1 n_2 \right) B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds,
\end{aligned}$$

since

$$\begin{aligned}
& \int u_1(v_1 - v_{*1}) n_1^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx \\
& = \int u_1(v_2 - v_{*2}) n_1 n_2 \chi_j B f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx = 0,
\end{aligned}$$

by an exchange of the variables v and v_* . Moreover, exchanging first the variables v and v_* ,

$$\begin{aligned}
& 2 \int_0^t \int v_1(v_2 - v_{*2}) n_1 n_2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& = \int_0^t \int (v_1 - v_{*1})(v_2 - v_{*2}) n_1 n_2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& \leq \frac{1}{\gamma'^2} \int_0^t \int (v_1 - v_{*1})^2 n_1^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& \quad + \frac{\gamma'^2}{4} \int_0^t \int (v_2 - v_{*2})^2 n_2^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \\
& \leq \frac{c_0}{\gamma'^2} (1+t) + \frac{\gamma'^2}{4} \int_0^t \int (v_2 - v_{*2})^2 n_2^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds.
\end{aligned}$$

It follows that

$$\int_0^t \int |v - v_*|^2 B \chi_j f f_* F_j(f') F_j(f'_*) dv dv_* d\theta dx ds \leq c'_0(1+t),$$

with c'_0 only depending on $\int f_0(x, v) dx dv$ and $\int |v|^2 f_0(x, v) dx dv$. This completes the proof of the lemma. \blacksquare

Lemma 3.3

There exist constants c'_1 and c'_2 only depending on $\int f_0(x, v) dx dv$ and on $\int |v|^2 f_0(x, v) dx dv$, so that

$$\int \sup_{0 \leq t \leq T} f_j^\sharp(t, x, v) dx dv < c'_1 + c'_2 T, \quad j \in \mathbb{N}^* \quad T > 0.$$

Proof of Lemma 3.3.

Denote f_j by f for simplicity. Since

$$f^\sharp(t, x, v) = f_0(x, v) + \int_0^t Q_j(f)(s, x + sv_1, v) ds,$$

it holds that

$$\sup_{0 \leq t \leq T} f^\sharp(t, x, v) \leq f_0(x, v) + \int_0^T Q_j^+(f)(t, x + tv_1, v) dt. \quad (3.7)$$

Integrating (3.7) with respect to (x, v) and using Lemma 3.2, gives

$$\begin{aligned} \int \sup_{0 \leq t \leq T} f^\sharp(t, x, v) dx dv &\leq \int f_0(x, v) dx dv + \frac{1}{\pi} \int_0^T \int B \chi_j \\ &\quad f(t, x + tv_1, v') f(t, x + tv_1, v'_*) F_j(f)(t, x + tv_1, v) F_j(f)(t, x + tv_1, v_*) dv dv_* d\theta dx dt \\ &\leq \int f_0(x, v) dx dv + \frac{1}{\gamma^2} \int_0^T \int B \chi_j |v - v_*|^2 \\ &\quad f(t, x, v') f(t, x, v'_*) F_j(f)(t, x, v) F_j(f)(t, x, v_*) dv dv_* d\theta dx dt \\ &\leq \int f_0(x, v) dx dv + \frac{C_1 + C_2 T}{\gamma^2}. \quad \blacksquare \end{aligned}$$

Lemma 3.4

Given $T > 0$ and $\delta_1 > 0$, there exist $\delta_2 > 0$ and $t_0 > 0$, only depending on $\int f_0(x, v) dx dv$ and on $\int |v|^2 f_0(x, v) dx dv$, such that for $t \leq T$

$$\sup_{x_0 \in [0, 1]} \int_{|x - x_0| < \delta_2} \sup_{t \leq s \leq t + t_0} f_j^\sharp(s, x, v) dx dv < \delta_1, \quad j \in \mathbb{N}^*.$$

Proof of Lemma 3.4.

Denote f_j by f for simplicity. For $t \leq s \leq t + t_0$ it holds,

$$\begin{aligned} f^\sharp(s, x, v) &= f^\sharp(t + t_0, x, v) - \int_s^{t+t_0} Q_j(f)(\tau, x + \tau v_1, v) d\tau \\ &\leq f^\sharp(t + t_0, x, v) + \int_s^{t+t_0} Q_j^-(f)(\tau, x + \tau v_1, v) d\tau. \end{aligned}$$

And so

$$\sup_{t \leq s \leq t + t_0} f^\sharp(s, x, v) \leq f^\sharp(t + t_0, x, v) + \int_t^{t+t_0} Q_j^-(f)(s, x + sv_1, v) ds.$$

Integrating with respect to (x, v) , using Lemma 3.2 and the bound $\frac{1}{\alpha}$ from above of f , gives

$$\begin{aligned}
& \int_{|x-x_0|<\delta_2} \sup_{t \leq s \leq t+t_0} f^\sharp(s, x, v) dx dv \\
& \leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0, x, v) dx dv \\
& + \frac{1}{\pi} \int_t^{t+t_0} \int B \chi_j f^\sharp(s, x, v) f(s, x + sv_1, v_*) F_j(f)(s, x + sv_1, v') F_j(f)(s, x + sv_1, v'_*) dv dv_* d\theta dx ds \\
& \leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0, x, v) dx dv + \frac{1}{\lambda^2} \int_t^{t+t_0} \int_{|v-v_*| \geq \lambda} B \chi_j |v - v_*|^2 f^\sharp(s, x, v) f(s, x + sv_1, v_*) \\
& F_j(f)(s, x + sv_1, v') F_j(f)(s, x + sv_1, v'_*) dv dv_* d\theta dx ds \\
& + c \int_t^{t+t_0} \int_{|v-v_*| < \lambda} B \chi_j f^\sharp(s, x, v) f(s, x + sv_1, v_*) dv dv_* d\theta dx ds \\
& \leq \int_{|x-x_0|<\delta_2} f^\sharp(t+t_0, x, v) dx dv + \frac{C_1 + C_2 T}{\lambda^2} + ct_0 \lambda^2 \int f^\sharp(t, x, v) dx dv \\
& \leq \frac{1}{\Lambda^2} \int v^2 f_0 dx dv + c\delta_2 \Lambda^2 + \frac{C_1 + C_2 T}{\lambda^2} + ct_0 \lambda^2 \int f_0(x, v) dx dv
\end{aligned}$$

Depending on δ_1 , suitably choosing Λ and then δ_2 , λ and then t_0 , the lemma follows. \blacksquare

The previous lemmas imply a t -dependent bound for the v -integral of f_j^\sharp only depending on $\int f_0(x, v) dx dv$ and on $\int |v|^2 f_0(x, v) dx dv$, as will now be proved.

Lemma 3.5

Given $T > 0$, the solution f_j of (3.1) satisfies

$$\int_{(t,x) \in [0,T] \times [0,1]} \sup_{v} f_j^\sharp(t, x, v) dv \leq c_1(T), \quad j \in \mathbb{N}^*,$$

where $c_1(T)$ only depends on T , $\int f_0(x, v) dx dv$ and $\int |v|^2 f_0(x, v) dx dv$.

Proof of Lemma 3.5.

For any $a, b \in \mathbb{R}$, denote by $I(a, b)$ the interval with end points a and b .

Denote by $E(x)$ the integer part of $x \in \mathbb{R}$, $E(x) \leq x < E(x) + 1$.

As in the proof of Lemma 3.3,

$$\begin{aligned}
\sup_{s \leq t} f^\sharp(s, x, v) & \leq f_0(x, v) + \int_0^t Q_j^+(f)(s, x + sv_1, v) ds = f_0(x, v) \\
& + \int_0^t \int B \chi_j f(s, x + sv_1, v') f(s, x + sv_1, v'_*) F_j(f)(s, x + sv_1, v) F_j(f)(s, x + sv_1, v'_*) dv_* d\omega ds \\
& \leq f_0(x, v) + cA,
\end{aligned} \tag{3.8}$$

where

$$A = \int_0^t \int B \chi_j \sup_{\tau \in [0, t]} f^\sharp(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0, t]} f^\sharp(\tau, x + s(v_1 - v'_1), v'_*) dv_* d\omega ds.$$

For θ outside of the angular cutoff (2.2), let n be the unit vector in the direction $v - v'$, and n_\perp the orthogonal unit vector in the direction $v - v'_*$. With e_1 a unit vector in the x -direction,

$$\max(|n \cdot e_1|, |n_\perp \cdot e_1|) \geq \frac{1}{\sqrt{2}}.$$

For $\delta_2 > 0$ that will be fixed later, split A into $A_1 + A_2 + A_3 + A_4$, where

$$\begin{aligned} A_1 &= \int_0^t \int_{|n \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| > \delta_2} B\chi_j \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_{*1}), v'_*) dv_* d\omega ds, \\ A_2 &= \int_0^t \int_{|n \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| < \delta_2} B\chi_j \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_{*1}), v'_*) dv_* d\omega ds, \\ A_3 &= \int_0^t \int_{|n_\perp \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| > \delta_2} B\chi_j \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_{*1}), v'_*) dv_* d\omega ds, \\ A_4 &= \int_0^t \int_{|n_\perp \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| < \delta_2} B\chi_j \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_1), v') \sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_{*1}), v'_*) dv_* d\omega ds. \end{aligned}$$

In A_1 and A_2 , bound the factor $\sup_{\tau \in [0, t]} f^\#(\tau, x + s(v_1 - v'_{*1}), v'_*)$ by its supremum over $x \in [0, 1]$, and make the change of variables

$$s \rightarrow y = x + s(v_1 - v'_1).$$

with Jacobian

$$\frac{Ds}{Dy} = \frac{1}{|v_1 - v'_1|} = \frac{1}{|v - v_*| |(n, \frac{v-v_*}{|v-v_*|})| |n_1|} \leq \frac{\sqrt{2}}{\gamma\gamma'}.$$

It holds that

$$A_1 \leq \int_{t|v_1 - v'_1| > \delta_2} \frac{B\chi_j}{|v_1 - v'_1|} \left(\int_{y \in I(x, x+t(v_1 - v'_1))} \sup_{\tau \in [0, t]} f^\#(\tau, y, v') dy \right) \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v'_*) dv_* d\omega,$$

and

$$A_2 \leq \frac{\sqrt{2}}{\gamma\gamma'} \int_{|n \cdot e_1| \geq \frac{1}{\sqrt{2}}, t|v_1 - v'_1| < \delta_2} B\chi_j \left(\int_{|y-x| < \delta_2} \sup_{\tau \in [0, t]} f^\#(\tau, y, v') dy \right) \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v'_*) dv_* d\omega.$$

Then, performing the change of variables $(v, v_*, \omega) \rightarrow (v', v'_*, -\omega)$,

$$\begin{aligned} & \int \sup_{x \in [0, 1]} A_1 dv \\ & \leq \int_{t|v_1 - v'_1| > \delta_2} \frac{B\chi_j}{|v_1 - v'_1|} \sup_{x \in [0, 1]} \left(\int_{y \in I(x, x+t(v'_1 - v_1))} \sup_{\tau \in [0, t]} f^\#(\tau, y, v) dy \right) \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v_*) dv dv_* d\omega, \end{aligned}$$

so that

$$\begin{aligned} & \int \sup_{x \in [0, 1]} A_1 dv \\ & \leq \int_{t|v_1 - v'_1| > \delta_2} \frac{B\chi_j}{|v_1 - v'_1|} \sup_{x \in [0, 1]} \left(\int_{y \in I(x, x+E(t(v'_1 - v_1) + 1))} \sup_{\tau \in [0, t]} f^\#(\tau, y, v) dy \right) \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v_*) dv dv_* d\omega \\ & = \int_{t|v_1 - v'_1| > \delta_2} \frac{B\chi_j}{|v_1 - v'_1|} |E(t(v'_1 - v_1) + 1)| \left(\int_0^1 \sup_{\tau \in [0, t]} f^\#(\tau, y, v) dy \right) \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v_*) dv dv_* d\omega \\ & \leq t(1 + \frac{1}{\delta_2}) \int B\chi_j \left(\int_0^1 \sup_{\tau \in [0, t]} f^\#(\tau, y, v) dy \right) \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v_*) dv dv_* d\omega \\ & \leq B_0 \pi t(1 + \frac{1}{\delta_2}) \int \sup_{\tau \in [0, t]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau, X) \in [0, t] \times [0, 1]} f^\#(\tau, X, v_*) dv_*. \end{aligned}$$

Apply Lemma 3.3, so that

$$\int \sup_{x \in [0,1]} A_1 dv \leq (c'_1 + c'_2 T) B_0 \pi t \left(1 + \frac{1}{\delta_2}\right) \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dv_*. \quad (3.9)$$

Moreover, performing the change of variables $(v, v_*, \omega) \rightarrow (v'_*, v', -\omega)$,

$$\int \sup_{x \in [0,1]} A_2 dv \leq \frac{B_0 \pi \sqrt{2}}{\gamma \gamma'} \sup_{x \in [0,1]} \left(\int_{|y-x| < \delta_2} \sup_{\tau \in [0,t]} f^\#(\tau, y, v_*) dy dv_* \right) \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v) dv.$$

Given $\delta_1 = \frac{\gamma \gamma'}{4 B_0 \pi \sqrt{2}}$, apply Lemma 3.4 with the corresponding δ_2 and t_0 , so that for $t \leq t_0$,

$$\int \sup_{x \in [0,1]} A_2 dv \leq \frac{1}{4} \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v) dv. \quad (3.10)$$

The terms A_3 and A_4 are treated similarly, with the change of variables $s \rightarrow y = x + s(v_1 - v'_{*1})$. Using (3.9)-(3.10) and the corresponding bounds obtained for A_3 and A_4 leads to

$$\begin{aligned} & \int \sup_{(s,x) \in [0,t] \times [0,1]} f^\#(s, x, v) dv \leq 2 \int \sup_{x \in [0,1]} f_0(x, v) dv \\ & + 4(c'_1 + c'_2 T) B_0 \pi t \left(1 + \frac{1}{\delta_2}\right) \int \sup_{(s,x) \in [0,t] \times [0,1]} f^\#(s, x, v) dv, \quad t \leq t_0. \end{aligned}$$

Hence for $t \leq \min(t_0, (8(c'_1 + c'_2 T) B_0 \pi^2 (1 + \frac{1}{\delta_2}))^{-1})$

$$\int \sup_{(s,x) \in [0,t] \times [0,1]} f^\#(s, x, v) dv \leq 4 \int \sup_{x \in [0,1]} f_0(x, v) dv.$$

Since c'_1 , c'_2 , and t_0 only depend on $\int (1 + |v|^2) f_0(x, v) dx dv$ and T , it follows that the argument can be repeated up to $t = T$. This completes the proof of the lemma. \blacksquare

4 Proof of the main theorem and the asymptotic behavior.

The following two preliminary lemmas are needed for the control of large velocities.

Lemma 4.1

Given $t > 0$, there is a constant $c_t > 0$ such that the solutions of (3.1) satisfy

$$\sup_{j \in \mathbb{N}^*} \int_0^1 \int_{|v| > \lambda} |v| \sup_{s \leq t} f_j^\#(s, x, v) dv dx \leq \frac{c_t}{\lambda}.$$

Proof of Lemma 4.1.

For convenience j is dropped from the notation f_j . As in Section 3,

$$\sup_{s \leq t} f^\#(s, x, v) \leq f_0(x, v) + \int_0^t Q_j^+(f)(s, x + sv_1, v) ds.$$

Integration with respect to (x, v) for $|v| > \lambda$, gives

$$\int_0^1 \int_{|v|>\lambda} |v| \sup_{s \leq t} f^\#(s, x, v) dv dx \leq \int \int_{|v|>\lambda} |v| f_0(x, v) dv dx + \int_0^t \int_{|v|>\lambda} B \chi_j |v| f(s, x + sv_1, v') f(s, x + sv_1, v'_*) F(f)(s, x + sv_1, v) F(f)(s, x + sv_1, v_*) dv dv_* d\omega ds.$$

Here in the last integral, either $|v'|$ or $|v'_*|$ is the largest and larger than $\frac{\lambda}{\sqrt{2}}$. The two cases are symmetric, and we discuss the case $|v'| \geq |v'_*|$. After a translation in x , the integrand is estimated from above by $c|v'| f^\#(s, x, v') \sup_{x \in [0,1], s \leq t} f^\#(s, x, v'_*)$. The change of variables $(v, v_*, \omega) \rightarrow (v', v'_*, -\omega)$, the integration over $(s, x, v, v_*, \omega) \in [0, t] \times [0, 1] \times \{v \in \mathbb{R}^2; |v| > \frac{\lambda}{\sqrt{2}}\} \times \mathbb{R}^2 \times S^1$ and Lemma 3.5 give the bound

$$\frac{c}{\lambda} \left(\int_0^t \int |v|^2 f^\#(s, x, v) dx dv ds \right) \left(\int \sup_{s \leq t, x \in [0,1]} f^\#(s, x, v_*) dv_* \right) \leq \frac{ctc_1(t)}{\lambda} \int |v|^2 f_0(x, v) dx dv.$$

The lemma follows. ■

Lemma 4.2

Given $t > 0$ and $\lambda > 2$, there is a constant $c'_t > 0$, such that the solutions f_j of (3.1) satisfy

$$\sup_{j \in \mathbb{N}^*} \int_{|v|>\lambda} \sup_{(s,x) \in [0,t] \times [0,1]} f_j^\#(s, x, v) dv \leq \frac{c'_t}{\sqrt{\lambda}}.$$

Proof of Lemma 4.2.

Take $\lambda > 2$. As above,

$$\int_{|v|>\lambda} \sup_{(s,x) \in [0,t] \times [0,1]} f^\#(s, x, v) dv \leq \int_{|v|>\lambda} \sup_{x \in [0,1]} f_0(x, v) dv + C, \quad (4.1)$$

where

$$C = c \int_{|v|>\lambda} \sup_{x \in [0,1]} \int_0^t \int B \chi_j f^\#(s, x + s(v_1 - v'_1), v') f^\#(s, x + s(v_1 - v'_1), v'_*) dv dv_* d\omega ds.$$

For v', v'_* outside of the angular cutoff (2.2), let n be the unit vector in the direction $v - v'$, and n_\perp the orthogonal unit vector in the direction $v - v'_*$. Let e_1 be a unit vector in the x -direction.

Split C as $C = \sum_{1 \leq i \leq 6} C_i$, where C_1 (resp. C_2, C_3) refers to integration on

$$\{(v_*, \omega); \quad n \cdot e_1 \geq \frac{1}{\sqrt{2}}, \quad |v'| \geq |v'_*|\},$$

$$(\text{resp. } \{(v_*, \omega); n \cdot e_1 \geq \sqrt{1 - \frac{1}{\lambda}}, |v'| \leq |v'_*|\}, \quad \{(v_*, \omega); n \cdot e_1 \in [\frac{1}{\sqrt{2}}, \sqrt{1 - \frac{1}{\lambda}}], |v'| \leq |v'_*|\}),$$

and analogously for C_i , $4 \leq i \leq 6$, with n replaced by n_\perp . By symmetry, C_i , $4 \leq i \leq 6$ can be treated as C_i , $1 \leq i \leq 3$, so we only discuss the control of C_i , $1 \leq i \leq 3$.

By the change of variables $(v, v_*, \omega) \rightarrow (v', v'_*, -\omega)$, and noticing that $|v'| \geq \frac{\lambda}{\sqrt{2}}$ in the domain of integration of C_1 , it holds that

$$\begin{aligned} C_1 &\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x \in [0,1]} \int_0^t \int_{n \cdot e_1 \geq \frac{1}{\sqrt{2}}} B \chi_j f^\#(s, x + s(v'_1 - v_1), v) f^\#(s, x + s(v'_1 - v_1), v_*) dv_* d\omega ds dv \\ &\leq \int_{|v|>\frac{\lambda}{\sqrt{2}}} \sup_{x \in [0,1]} \int_0^t \int_{n \cdot e_1 \geq \frac{1}{\sqrt{2}}} B \chi_j \sup_{\tau \in [0,t]} f^\#(\tau, x + s(v'_1 - v_1), v) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dv_* d\omega ds dv. \end{aligned}$$

With the change of variables $s \rightarrow y = x + s(v'_1 - v_1)$,

$$\begin{aligned} C_1 &\leq \int_{|v| > \frac{\lambda}{\sqrt{2}}} \sup_{x \in [0,1]} \int_{n \cdot e_1 \geq \frac{1}{\sqrt{2}}} \int_{y \in I(x, x+t(v'_1-v_1))} \frac{B\chi_j}{|v'_1 - v_1|} \sup_{\tau \in [0,t]} f^\#(\tau, y, v) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dy dv_* d\omega dv \\ &\leq \int_{|v| > \frac{\lambda}{\sqrt{2}}} \int_{n \cdot e_1 \geq \frac{1}{\sqrt{2}}} \frac{|E(t(v'_1 - v_1)) + 1|}{|v'_1 - v_1|} \int_0^1 B\chi_j \sup_{\tau \in [0,t]} f^\#(\tau, y, v) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dy dv_* d\omega dv. \end{aligned}$$

Moreover,

$$|E(t(v'_1 - v_1)) + 1| \leq t|v'_1 - v_1| + 1 \leq \left(t + \frac{\sqrt{2}}{\gamma\gamma'}\right)|v'_1 - v_1|.$$

Consequently,

$$\begin{aligned} C_1 &\leq c(t+1) \int_0^1 \int_{|v| > \frac{\lambda}{\sqrt{2}}} \sup_{\tau \in [0,t]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dv_* \\ &\leq \frac{c(t+1)}{\lambda} \int_0^1 \int_{|v| > \frac{\lambda}{\sqrt{2}}} |v| \sup_{\tau \in [0,t]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dv_*. \end{aligned}$$

By Lemma 3.5 and Lemma 4.1,

$$C_1 \leq \frac{c}{\lambda^2} (t+1) c_t c_1(t).$$

Moreover,

$$\begin{aligned} C_2 &\leq \int_{|v'| > \lambda, |v_*| > |v|, n \cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} \frac{B\chi_j}{|v'_1 - v_1|} \\ &\quad \sup_{x \in [0,1]} \int_{y \in I(x, x+t(v'_1-v_1))} \sup_{\tau \in [0,t]} f^\#(\tau, y, v) \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dy dv dv_* d\omega \\ &\leq c(t+1) \int_{n \cdot e_1 \geq \sqrt{1-\frac{1}{\lambda}}} d\omega \int \sup_{\tau \in [0,t]} f^\#(\tau, y, v) dy dv \int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v_*) dv_* \\ &\leq \frac{c}{\sqrt{\lambda}} (t+1)^2 c_1(t), \end{aligned}$$

by Lemma 3.3 and Lemma 3.5. Finally,

$$\begin{aligned} C_3 &\leq \int_{|v_*| > \frac{\lambda}{\sqrt{2}}, \frac{1}{\sqrt{\lambda}} \leq n_\perp \cdot e_1 \leq \frac{1}{\sqrt{2}}} \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v) \frac{B\chi_j}{|v'_1 - v_{*1}|} \\ &\quad \sup_{x \in [0,1]} \left(\int_{y \in I(x, x+t(v'_1-v_{*1}))} \sup_{\tau \in [0,t]} f^\#(\tau, y, v_*) dy \right) dv dv_* d\omega \\ &\leq c(t+1) \sqrt{\lambda} \left(\int \sup_{(\tau, X) \in [0,t] \times [0,1]} f^\#(\tau, X, v) dv \right) \left(\int_{|v_*| > \frac{\lambda}{\sqrt{2}}} \sup_{\tau \in [0,t]} f^\#(\tau, y, v_*) dy dv_* \right). \end{aligned}$$

By Lemma 3.5,

$$C_3 \leq c(t+1) \sqrt{\lambda} c_1(t) \int_{|v_*| > \frac{\lambda}{\sqrt{2}}} \sup_{\tau \in [0,t]} f^\#(\tau, y, v_*) dy dv_*,$$

and so by Lemma 4.1,

$$C_3 \leq \frac{c}{\sqrt{\lambda}}(t+1)c_1(t)c_t.$$

The lemma follows. ■

Using the previous lemmas, the results in Section 3, and an initial layer analysis, the main result of the paper follows.

Proof of Theorem 2.1.

For any $T > 0$, we shall prove the convergence in $C([0, T]; L^1([0, T] \times \mathbb{R}^2))$ of the sequence (f_j) to a solution f of (2.5). Denote by

$$\tilde{\nu}_j(f) := \int B\chi_j f' f'_* F_j(f_*) dv_* d\omega, \quad \nu_j(f) := \int B\chi_j f_* F_j(f') F_j(f'_*) dv_* d\omega,$$

so that

$$Q_j(f) = F_j(f)\tilde{\nu}_j(f) - f\nu_j(f).$$

Consider

$$\nu_j(f_j)^\sharp(t, x, v) = \int B\chi_j f_j(t, x + tv_1, v_*) F_j(f_j(t, x + tv_1, v')) F_j(f_j(t, x + tv_1, v'_*)) dv_* d\omega.$$

With the angular cut-off (2.2), $v_* \rightarrow v'$ and $v_* \rightarrow v'_*$ are changes of variables. Indeed, if the polar coordinates of $v_* - v$ are (r_*, φ) and θ is the angle between $v_* - v$ and n , then the polar coordinates of $v' - v$ (resp. $v'_* - v$) are $(|r_* \cos \theta|, \varphi + \theta)$ (resp. $(|r_* \sin \theta|, \varphi + \theta + \frac{\pi}{2})$). It follows from the angular cut-off (2.2), that the Jacobians $\frac{Dv_*}{Dv'} = \frac{1}{|\cos \theta|}$ (resp. $\frac{Dv_*}{Dv'_*} = \frac{1}{|\sin \theta|}$) are bounded. Using these changes of variables and Lemma 3.5, for ω outside the integration cut-off, the measure of the set

$$Z_{(j,t,x,v,\omega)} := \{v_*; f(t, x + tv_1, v') > \frac{1}{2} \quad \text{or} \quad f(t, x + tv_1, v'_*) > \frac{1}{2}\} \quad (4.2)$$

is uniformly bounded with respect to (x, v, ω) , $t \leq T$, and $j \in \mathbb{N}^*$. Take j_T so large that πj_T^2 is at least eight times this uniform bound. Notice that here j_T only depends on T and $\int (1 + v^2) f_0(x, v) dx dv$. Using the exponential form for the solution, one gets using Lemma 3.5 that

$$f_j^\sharp(t, x, v_*) \geq c_{1T} f_0(x, v_*) > 0, \quad j \geq j_T, \quad t \leq T, \quad (4.3)$$

with c_{1T} independent of $j \geq j_T$. It follows from (4.3) and the third assumption in (2.4) that

$$\nu_j(f_j)^\sharp(t, x, v) > c_{2T} > 0, \quad (t, x, v) \in [0, T] \times [0, 1] \times \{v \in \mathbb{R}^2; |v| \leq j\}, \quad (4.4)$$

uniformly with respect to $j \geq j_T$, and with c_{2T} only depending on T and f_0 .

Using again the $v_* \rightarrow v'$ change of variables together with Lemma 3.5, one obtains that for some constant $c_{3T} > 0$,

$$\tilde{\nu}_j^\sharp(f_j)(t, x, v) \leq c_{3T}, \quad j \geq j_T, \quad (t, x, v) \in [0, T] \times [0, 1] \times \{v \in \mathbb{R}^2; |v| \leq j\}.$$

The functions defined on $]0, \frac{1}{\alpha}]$ by $x \rightarrow \frac{F_j(x)}{x}$ are uniformly bounded from above with respect to j by

$$x \rightarrow c\alpha^{\alpha-1} \frac{(1-\alpha x)^\alpha}{x},$$

that is continuous and decreasing to zero at $x = \frac{1}{\alpha}$. Hence there is $\mu \in]0, \frac{1}{\alpha}[$ such that

$$x \in [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}] \quad \text{implies} \quad \frac{F_j(x)}{x} \leq \frac{c_{2T}}{4c_{3T}}, \quad j \geq j_T.$$

Consequently, for $j \geq j_T$,

$$\begin{aligned} f_j^\#(t, x, v) \in [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}] &\Rightarrow D_t f_j^\#(t, x, v) = (F_j(f_j^\#) \tilde{\nu}_j^\# - \frac{1}{2} f_j^\# \nu_j^\#)(t, x, v) - \frac{1}{2} f_j^\# \nu_j^\#(t, x, v) \\ &< -\frac{1}{2} f_j^\# \nu_j^\#(t, x, v) \\ &< -\frac{1}{2} (\frac{1}{\alpha} - \mu) c_{2T} := -b_1. \end{aligned}$$

This gives a maximum time $t_1 = \frac{\mu}{b_1}$ for $f_j^\#$ to reach $\frac{1}{\alpha} - \mu$ from an initial value $f_0(x, v) \in]\frac{1}{\alpha} - \mu, \frac{1}{\alpha}]$. On this time interval $D_t f_j^\# \leq -b_1$. If $t_1 \geq T$, then at $t = T$ the value of $f_j^\#$ is bounded from above by $\frac{1}{\alpha} - b_1 T := \frac{1}{\alpha} - \mu'$ with $0 < \mu' \leq \mu$. Take $t_m = \min(t_1, T)$, and from now on $\mu = t_m b_1$. For any (x, v) , if $f_j(0, x, v) < \frac{1}{\alpha} - \mu$ were to reach $\frac{1}{\alpha} - \mu$ at (t, x, v) with $t \leq t_m$, then $D_t f_j^\#(t, x, v) \leq -b_1$, which excludes such a possibility. It follows that $f_j \leq \frac{1}{\alpha} - \mu$ everywhere for $t \in [t_m, T]$, and that

$$f_j^\#(t, x, v) \leq \frac{1}{\alpha} - b_1 t. \quad (4.5)$$

for $t \in [0, t_m]$. The previous estimates leading to the definition of t_m are independent of $j \geq j_T$.

Let us prove that (f_j) converges in $L^1([0, T] \times [0, 1] \times \mathbb{R}^2)$ when $j \rightarrow \infty$.

We shall prove that given $\beta > 0$, there exists $a \geq \max\{1, j_T\}$, so that

$$\sup_{t \in [0, T]} \int |g_j(t, x, v)| dx dv < \beta, \quad j > a, \quad (4.6)$$

where $g_j = f_j - f_a$. The function g_j satisfies the equation

$$\begin{aligned} \partial_t g_j + p_1 \partial_x g_j &= \int (\chi_j - \chi_a) B \left(f_j' f_{j*}' F_j(f_j) F_j(f_{j*}) - f_j f_{j*} F_j(f_j') F_j(f_{j*}') \right) dv_* d\omega \\ &+ \int \chi_a B (f_j' f_{j*}' - f_a' f_{a*}') F_j(f_j) F_j(f_{j*}) dv_* d\omega \\ &- \int \chi_a B (f_j f_{j*} - f_a f_{a*}) F_j(f_j') F_j(f_{j*}') dv_* d\omega \\ &+ \int \chi_a B f_a' f_{a*}' \left(F_j(f_{j*}) (F_j(f_j) - F_j(f_a)) + F_a(f_a) (F_j(f_{j*}) - F_j(f_{a*})) \right) dv_* d\omega \\ &+ \int \chi_a B f_a' f_{a*}' \left(F_j(f_{j*}) (F_j(f_a) - F_a(f_a)) + F_a(f_a) (F_j(f_{a*}) - F_a(f_{a*})) \right) dv_* d\omega \\ &- \int \chi_a B f_a f_{a*} \left(F_j(f_{j*}') (F_j(f_j') - F_j(f_a')) + F_a(f_a') (F_j(f_{j*}') - F_j(f_{a*}')) \right) dv_* d\omega \\ &- \int \chi_a B f_a f_{a*} \left(F_j(f_{j*}') (F_j(f_a') - F_a(f_a')) + F_a(f_a') (F_j(f_{a*}') - F_a(f_{a*}')) \right) dv_* d\omega. \quad (4.7) \end{aligned}$$

Moreover, using Lemma 3.5

$$\begin{aligned}
& \int (\chi_j - \chi_a) B \left(f'_j f'_{j*} F_j(f_j) F_j(f_{j*}) + f_j f_{j*} F_j(f'_j) F_j(f'_{j*}) \right) dx dv dv_* d\omega \\
& \leq c \int_{|v| > \frac{a}{\sqrt{2}}} f_j(t, x, v) dx dv \\
& \leq \frac{c}{a^2} \text{ by the conservation of energy of } f_j, \\
& \int \chi_a B |f_j f_{j*} - f_a f_{a*}| F_j(f'_j) F_j(f'_{j*}) dx dv dv_* d\omega \\
& \leq c \left(\int \sup_{(t,x) \in [0,T] \times [0,1]} f_j^\sharp(t, x, v) dv + \int \sup_{(t,x) \in [0,T] \times [0,1]} f_a^\sharp(t, x, v) dv \right) \int |(f_j^\sharp - f_a^\sharp)(t, x, v)| dx dv \\
& \leq c \int |(f_j^\sharp - f_a^\sharp)(t, x, v)| dx dv \quad \text{by Lemma 3.5.}
\end{aligned}$$

Next,

$$\begin{aligned}
& \int \chi_a B \left(f'_a f'_{a*} F_j(f_{j*}) |F_j(f_a) - F_a(f_a)| \right)^\sharp dx dv dv_* d\omega \\
& = \int \chi_a B f'_a f'_{a*} F_j(f_{j*}) (1 - \alpha f_a) (1 + (1 - \alpha) f_a)^{1-\alpha} \left| \left(\frac{1}{j} + 1 - \alpha f_a \right)^{\alpha-1} - \left(\frac{1}{a} + 1 - \alpha f_a \right)^{\alpha-1} \right| dx dv dv_* d\omega.
\end{aligned}$$

By Lemma 3.3 and Lemma 3.5, this integral restricted to the set where $1 - \alpha f_a(t, x, v) \leq \frac{2}{a}$, hence where

$$(1 - \alpha f_a) \left| \left(\frac{1}{j} + 1 - \alpha f_a \right)^{\alpha-1} - \left(\frac{1}{a} + 1 - \alpha f_a \right)^{\alpha-1} \right| \leq 2(1 - \alpha f_a)^\alpha \leq \frac{2^{\alpha+1}}{a^\alpha},$$

is bounded by $\frac{c}{a^\alpha}$ for some constant $c > 0$.

For the remaining domain of integration where $1 - \alpha f_a(t, x, v) \geq \frac{2}{a}$, it holds

$$\begin{aligned}
|F_j(f_a) - F_a(f_a)| & \leq c(1 - \alpha f_a)^\alpha \left| \left(\frac{1}{j(1 - \alpha f_a)} + 1 \right)^{\alpha-1} - \left(\frac{1}{a(1 - \alpha f_a)} + 1 \right)^{\alpha-1} \right| \\
& = c \left(\frac{1}{j} - \frac{1}{a} \right) (1 - \alpha f_a)^{\alpha-1} \lambda^{\alpha-2} \quad \text{where } \lambda \in [1, \frac{3}{2}] \\
& \leq \frac{2^{\alpha-1} c}{a^\alpha}.
\end{aligned}$$

And so,

$$\int \chi_a B \left(f'_a f'_{a*} F_j(f_{j*}) |F_j(f_a) - F_a(f_a)| \right)^\sharp dx dv dv_* d\omega \leq \frac{c}{a^\alpha}.$$

Finally

$$\int \chi_a B \left(f'_a f'_{a*} F_j(f_{j*}) |F_j(f_j) - F_j(f_a)| \right)^\sharp(t, x, v) dx dv dv_* d\omega \leq c \int |F_j(f_j) - F_j(f_a)|^\sharp(t, x, v) dx dv.$$

Split the (x, v) -domain of integration of the latest integral into

$$\begin{aligned}
D_1 &:= \{(x, v); (f_j^\sharp(t, x, v), f_a^\sharp(t, x, v)) \in [0, \frac{1}{\alpha} - \mu]^2\}, \\
D_2 &:= \{(x, v); (f_j^\sharp(t, x, v), f_a^\sharp(t, x, v)) \in [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}]^2\}, \\
D_3 &:= \{(x, v); (f_j^\sharp, f_a^\sharp)(t, x, v) \in [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}] \times [0, \frac{1}{\alpha} - \mu] \text{ or } (f_j^\sharp, f_a^\sharp)(t, x, v) \in [0, \frac{1}{\alpha} - \mu] \times [\frac{1}{\alpha} - \mu, \frac{1}{\alpha}]\}.
\end{aligned}$$

It holds that

$$\begin{aligned} \int_{D_1} |F_j(f_j) - F_j(f_a)|^\sharp(t, x, v) dx dv &\leq c(\alpha\mu)^{\alpha-1} \int_{D_1} |g_j^\sharp(t, x, v)| dx dv, \\ \int_{D_2} |F_j(f_j) - F_j(f_a)|^\sharp(t, x, v) dx dv &\leq ct^{\alpha-1} \int_{D_2} |g_j^\sharp(t, x, v)| dx dv, \quad \text{by (4.5),} \\ \int_{D_3} |F_j(f_j) - F_j(f_a)|^\sharp(t, x, v) dx dv &\leq c((\alpha\mu)^{\alpha-1} + t^{\alpha-1}) \int_{D_3} |g_j^\sharp(t, x, v)| dx dv. \end{aligned}$$

The remaining terms to the right in (4.7) are of the same types as the ones just estimated. Consequently,

$$\frac{d}{dt} \int |g_j^\sharp(t, x, v)| dx dv \leq \frac{c}{a^\alpha} + c(1 + t^{\alpha-1}) \left(\int |g_j^\sharp(t, x, v)| dx dv \right).$$

And so,

$$\int |g_j^\sharp(t, x, v)| dx dv \leq \left(\int_{|v|>a} f_0(x, v) dx dv + \frac{cT}{a^\alpha} \right) e^{c(T + \frac{T^\alpha}{\alpha})},$$

which tends to zero when $a \rightarrow +\infty$, uniformly w.r.t. $j \geq a$. This proves that $(f_j)_{j \in \mathbb{N}^*}$ is a Cauchy sequence in $L^1([0, T] \times [0, 1] \times \mathbb{R}^2)$ and ends the proof of the existence of a solution f to (2.5).

One can similarly prove that the solution is unique and stable. The energy is non-increasing. The conservation of mass and first momentum of f follow from the boundedness of the total energy.

Energy conservation will follow if the energy is non-decreasing. Taking $\psi_\epsilon = \frac{|v^2|}{1+\epsilon|v|^2}$ as approximation for $|v|^2$, it is enough to bound

$$\int Q(f, f)(t, x, v) \psi_\epsilon(v) dx dv = \int B \psi_\epsilon \left(f' f'_* F(f) F(f_*) - f f_* F(f') F(f'_*) \right) dx dv dv_* d\omega$$

from below by zero in the limit $\epsilon \rightarrow 0$. Now [12]

$$\begin{aligned} \int Q(f, f) \psi_\epsilon dx dv &= \frac{1}{2} \int B f f_* F(f') F(f'_*) \left(\psi_\epsilon(v') + \psi_\epsilon(v'_*) - \psi_\epsilon(v) - \psi_\epsilon(v_*) \right) dx dv dv_* d\omega \\ &\geq - \int B f f_* F(f') F(f'_*) \frac{\epsilon |v|^2 |v_*|^2}{(1 + \epsilon |v|^2)(1 + \epsilon |v_*|^2)} dx dv dv_* d\omega. \end{aligned}$$

The previous line, with the integral taken over a bounded set in (v, v_*) , converges to zero when $\epsilon \rightarrow 0$. In integrating over $|v|^2 + |v_*|^2 \geq 2\lambda^2$, there is symmetry between the subset of the domain with $|v|^2 > \lambda^2$ and the one with $|v_*|^2 > \lambda^2$. We discuss the first sub-domain, for which the integral in the last line is bounded from below by

$$-c \int |v_*|^2 f(t, x, v_*) dx dv_* \int_{|v| \geq \lambda} B \sup_{(s, x) \in [0, t] \times [0, 1]} f^\sharp(s, x, v) dv d\omega \geq -c \int_{|v| \geq \lambda} \sup_{0 \leq s, x \in [0, 1]} f^\sharp(s, x, v) dv.$$

It follows from Lemma 4.2 that the right hand side tends to zero when $\lambda \rightarrow \infty$. This implies that the energy is non-decreasing, and bounded from below by its initial value. That completes the proof of the theorem. \blacksquare

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